Extension of Berenger's Absorbing Boundary Conditions to Match Dielectric Anisotropic Media

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***Abstract*— The authors propose an extension of Berenger's perfectly matched layer (PML) absorbing boundary conditions (ABC's) to achieve a perfect matching of waves propagating in anisotropic media. Although the procedure to obtain the matching conditions is valid for any kind of anisotropic material, it has been validated with a lossless two-dimensional uniaxial medium, in which the optical axis is not contained in its plane section. The finite difference time domain method, with an alternative scheme for anisotropic media, is used to simulate the problem and to obtain the numerical reflection coefficient.**

***Index Terms*—Absorbing boundary conditions, anisotropic media, FDTD methods, perfect matching layers.**

I. Introduction

THE application of finite methods to the open problem of propagation of electromagnetic waves in unbounded media requires the use of the so-called absorbing boundary conditions (ABC's) in order to truncate the computational region. Berenger's perfectly matched layer (PML) ABC's [1], [2] is an outstanding contribution which greatly reduces the reflection coefficients obtained with previous methods when applied to isotropic media.

The propagation of waves in anisotropic materials, which are commonly used in microwave components, microstrip waveguides, and fiber optics, is an area of current interest. Unfortunately ABC's based on one-way operators cannot be applied to anisotropic materials. The authors have recently proven [3] that Berenger's ABC's cannot provide a general matching condition capable of simultaneously absorbing the ordinary and extraordinary modes propagating in a general anisotropic medium; its application was then limited to the uncoupled problem of propagation in a two-dimensional (2-D) anisotropic medium with the optical axis contained in its plane. In this letter we propose a method to effectively extend Berenger's PML to achieve a perfect matching of arbitrary waves propagating in anisotropic media, with the optical axis arbitrarily oriented. Although the procedure to obtain the matching conditions is valid for any kind of anisotropic material, in this letter we limit ourselves, for simplicity, to the general 2-D problem of absorbing a wave propagating in a lossless anisotropic medium (LAM). Results are presented for the simultaneous absorption of an ordinary and an extraordinary mode propagating in a 2-D uniaxial LAM in which the optical axis is not contained in its plane. The finite-difference time-domain (FDTD) method, with an alternative scheme for anisotropic media [4], [5], is used to simulate the problem and to obtain the numerical reflection coefficient.

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II. The PML Problem

The problem of obtaining a PML to absorb incoming waves from a given medium can be summarized in a simple manner. Suppose that a plane wave propagates in a lossless medium m in the direction $\hat{n}$ with phase velocity v

$$\vec{\Psi}_m(\vec{r}, t) = \vec{\Psi}_{mo} e^{jw\left(t - \frac{\hat{n}\cdot\vec{r}}{v}\right)} \tag{1}$$

and strikes a PML interface at the plane $z = 0$. This PML must provide a finite reflectionless region for all angles of incidence. Consequently, there must be perfect matching between the original and the PML media, and the transmitted wave must be attenuated in the latter. This is attained by choosing a complex propagation factor $\vec{\alpha}$ such that the transmitted wave in the matching layer l has the form

$$\vec{\Psi}_l(\vec{r}, t) = \vec{\Psi}_{lo} e^{jw(t - \vec{\alpha}\cdot\vec{r})}. \tag{2}$$

Nonreflection at the interface is ensured by imposing the following two conditions.

1) The relation between the field components in the incident medium must be the same as in the PML.
2) The exponentials of (1) and (2) must take the same values at the interface $z = 0$

$$\left(e^{j\omega\frac{\hat{n}\cdot\vec{r}}{v}} = e^{j\omega\vec{\alpha}\cdot\vec{r}}\right)_{z=0}. \tag{3}$$

Notice that this condition is satisfied independently of α_z if

$$\alpha_x = \frac{n_x}{v}, \quad \alpha_y = \frac{n_y}{v}. \tag{4}$$

III. Matching Anisotropic Media

In order to impose the above two conditions, in this section we will first write Maxwell's equations for the anisotropic medium, and then we will propose an extended PML (EPML) that achieves the matching.

Let us consider a LAM for which Maxwell's rotational equations in the frequency domain are

$$-j\omega\mu\vec{H} = \nabla\times\vec{E}, \quad j\omega\tilde{\varepsilon}\cdot\vec{E} = \nabla\times\vec{H} \tag{5}$$

where $\tilde{\varepsilon} = (\varepsilon_{ij})$ is the real and symmetrical dielectric tensor of the medium.

The relation between the components of the plane wave of (1) is

$$\mu \vec{H} = \frac{\hat{n}}{v} \times \vec{E}, \quad -\tilde{\varepsilon} \cdot \vec{E} = \frac{\hat{n}}{v} \times \vec{H}. \tag{6}$$

Let the wave propagation take place in the XZ plane of a 2-D LAM with the optical axis arbitrarily oriented, so that $n_y = 0$. Equations (5) then become

$$j\omega\mu H_x = \frac{\partial E_y}{\partial z} \tag{7a}$$
$$j\omega\mu H_z = -\frac{\partial E_y}{\partial x} \tag{7b}$$
$$j\omega(\varepsilon_{xx}E_x + \varepsilon_{xy}E_y + \varepsilon_{xz}E_z) = -\frac{\partial H_y}{\partial z} \tag{7c}$$
$$j\omega(\varepsilon_{xz}E_x + \varepsilon_{yz}E_y + \varepsilon_{zz}E_z) = \frac{\partial H_y}{\partial x} \tag{7d}$$
$$j\omega\mu H_y = \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \tag{7e}$$
$$j\omega(\varepsilon_{xy}E_x + \varepsilon_{yy}E_y + \varepsilon_{yz}E_z) = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \tag{7f}$$

and relation (6) becomes

$$\mu H_x = -\frac{n_z}{v}E_y \tag{8a}$$
$$\mu H_z = \frac{n_x}{v}E_y \tag{8b}$$
$$\varepsilon_{xx}E_x + \varepsilon_{xy}E_y + \varepsilon_{xz}E_z = \frac{n_z}{v}H_y \tag{8c}$$
$$\varepsilon_{xz}E_x + \varepsilon_{yz}E_y + \varepsilon_{zz}E_z = -\frac{n_x}{v}H_y \tag{8d}$$
$$\mu H_y = \frac{n_z}{v}E_x - \frac{n_x}{v}E_z \tag{8e}$$
$$\varepsilon_{xy}E_x + \varepsilon_{yy}E_y + \varepsilon_{yz}E_z = \frac{n_x}{v}H_z - \frac{n_z}{v}H_x. \tag{8f}$$

To obtain the characteristics of the EPML that generalizes Berenger's method to anisotropic media, the transversal field components are split into two subcomponents, $E_y = E_{yx} + E_{yz}$ and $H_y = H_{yx} + H_{yz}$ and (7e) and (7f) into (9e), (9f) and (9g), (9h), respectively. Furthermore, the real constants in (7) are allowed to take complex values μ^x, μ^z, ε^x_{ij} and ε^z_{ij}.

$$j\omega\mu^z H_x = \frac{\partial E_y}{\partial z} \tag{9a}$$
$$j\omega\mu^x H_z = -\frac{\partial E_y}{\partial x} \tag{9b}$$
$$j\omega\left(\varepsilon^z_{xx}E_x + \varepsilon^z_{xy}E_y + \varepsilon^z_{xz}E_z\right) = -\frac{\partial H_y}{\partial z} \tag{9c}$$
$$j\omega\left(\varepsilon^x_{xz}E_x + \varepsilon^x_{yz}E_y + \varepsilon^x_{zz}E_z\right) = \frac{\partial H_y}{\partial x} \tag{9d}$$
$$j\omega\mu^x H_{yx} = \frac{\partial E_z}{\partial x} \tag{9e}$$
$$j\omega\mu^z H_{yz} = -\frac{\partial E_x}{\partial z} \tag{9f}$$
$$j\omega\left(\frac{1}{2}\varepsilon^x_{xy}E_x + \varepsilon^x_{yy}E_{yx} + \frac{1}{2}\varepsilon^x_{yz}E_z\right) = -\frac{\partial H_z}{\partial x} \tag{9g}$$
$$j\omega\left(\frac{1}{2}\varepsilon^z_{xy}E_x + \varepsilon^z_{yy}E_{yz} + \frac{1}{2}\varepsilon^z_{yz}E_z\right) = \frac{\partial H_x}{\partial z}. \tag{9h}$$

To derive the relation between the field components in the EPML, (2) must be inserted into the above equations, obtaining

$$\mu^z H_x = -\alpha_z E_y, \quad \mu^x H_z = \alpha_x E_y \tag{10a}$$

$$\mu^z H_{yz} = \alpha_z E_x, \quad \mu^x H_{yx} = -\alpha_x E_z, \quad H_y = H_{yx} + H_{yz} \tag{10b}$$

$$\varepsilon^z_{xx}E_x + \varepsilon^z_{xy}E_y + \varepsilon^z_{xz}E_z = \alpha_z H_y \tag{10c}$$
$$\varepsilon^x_{xz}E_x + \varepsilon^x_{yz}E_y + \varepsilon^x_{zz}E_z = -\alpha_x H_y \tag{10d}$$
$$\frac{1}{2}\varepsilon^x_{xy}E_x + \varepsilon^x_{yy}E_{yx} + \frac{1}{2}\varepsilon^x_{yz}E_z = \alpha_x H_z \tag{10e}$$
$$\frac{1}{2}\varepsilon^z_{xy}E_x + \varepsilon^z_{yy}E_{yz} + \frac{1}{2}\varepsilon^z_{yz}E_z = -\alpha_z H_x \tag{10f}$$
$$E_y = E_{yx} + E_{yz}. \tag{10g}$$

Identifying relations (8) and (10), condition 1) of Section II is satisfied for all μ^x and μ^z if

$$\alpha_x = \frac{\mu^x}{\mu}\frac{n_x}{v}, \quad \alpha_z = \frac{\mu^z}{\mu}\frac{n_z}{v} \tag{11a}$$
$$\varepsilon^x_{ij} = \frac{\mu^x}{\mu}\varepsilon_{ij}, \quad \varepsilon^z_{ij} = \frac{\mu^z}{\mu}\varepsilon_{ij} \tag{11b}$$

and condition 2) [see (4)] is satisfied if $\mu^x = \mu$. Given that μ^z can be arbitrarily chosen, it is taken such that the transmitted wave is attenuated within the EPML. Specifically we take[1]

$$\mu^z = \mu\left(1 - j\frac{\sigma^*_z}{\omega\mu}\right), \quad \sigma^*_z > 0. \tag{12}$$

Then a wave propagating in the EPML has the form

$$\vec{\Psi}_l(\vec{r}, t) = \vec{\Psi}_{lo}\, e^{j\omega\left(t - \frac{n_x x + n_z z}{v}\right)}\, e^{-\frac{\sigma^*_z n_z z}{\mu v}} \tag{13}$$

which after crossing the EPML is reflected by perfectly conducting conditions which end the domain. We choose for σ^*_z a profile, as in [1], of the form

$$\sigma^*_z = \sigma^*_{mz}\left(\frac{\rho}{\delta}\right)^n \tag{14}$$

with σ^*_{mz} a maximum value for σ^*_z, ρ the distance from the beginning of the EPML and δ the total depth of the EPML medium. The theoretical reflection coefficient, as a function of the incidence angle θ is then

$$R(\theta) = e^{-\frac{2}{n+1}\frac{\sigma^*_{mz}\delta\cos\theta}{\mu v}} \tag{15}$$

[1] Berenger defines an electric conductivity in the isotropic case that fulfills $\sigma_z = \frac{\varepsilon}{\mu}\sigma^*_z$. This could be translated to the present anisotropic case by defining a set of electric conductivities of the form $\sigma_{z_{ij}} = \frac{\varepsilon_{ij}}{\mu}\sigma^*_z$.

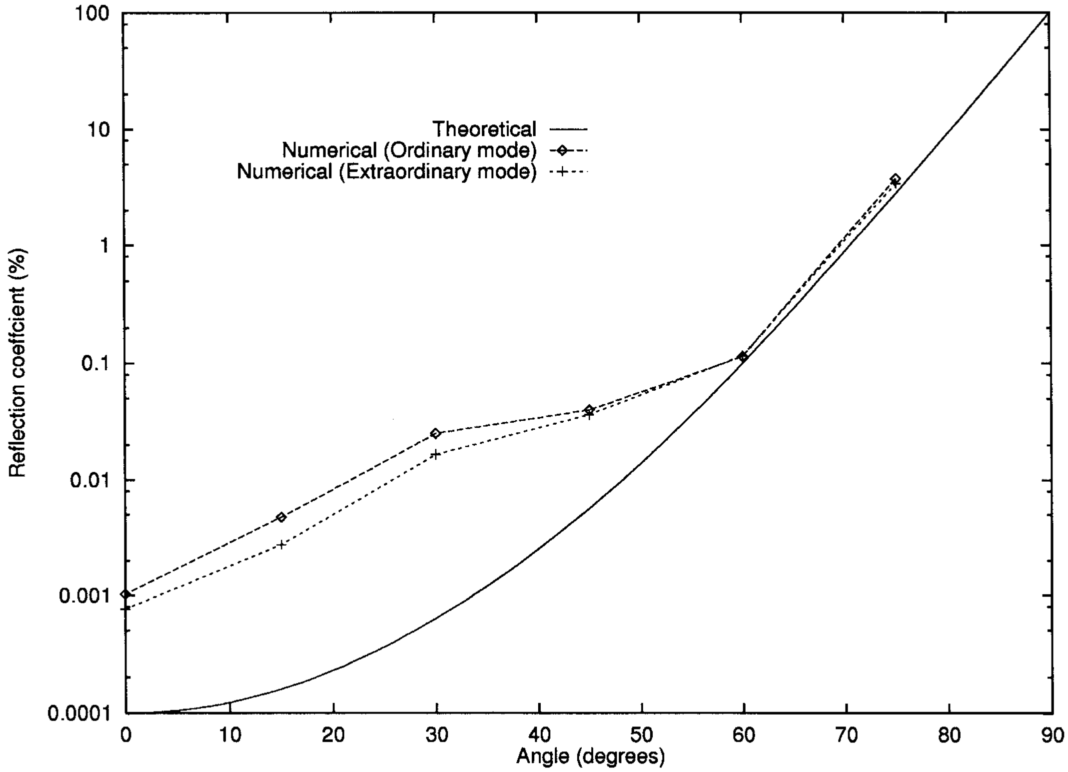

Fig. 1. Reflection coefficient as a function of the incidence angle.

IV. Results and Conclusions

In summary, perfect matching has been attained in the two following steps.

1) Two sets of conditions are obtained: first, the ones derived from requiring the relations between the field components to be the same for the waves propagating in the EPML and for the waves propagating in the anisotropic medium; and secondly, the ones derived from the enforcement of the continuity of the exponentials [see (4)].
2) The characteristics of the EPML are chosen so that the waves propagating in it are attenuated.

We have numerically validated the above results by simulating the incidence with a EPML of monochromatic ordinary and extraordinary plane waves propagating in a uniaxial 2-D LAM lying in the XZ plane. This has been carried out using an alternative scheme of the FDTD method to handle anisotropic materials.

The dielectric tensor of the anisotropic material used in the numerical tests has, in its principal coordinate system, the following elements: $\varepsilon_x = \varepsilon_y = 2.272\varepsilon_0$, $\varepsilon_z = 2.187\varepsilon_0$. The optical axis is contained in the YZ plane forming an angle $\Phi = 30^\circ$ with the Z-axis. The EPML medium was chosen with a depth of eight spatial cells, with a quadratic variation of σ_z^* [$n = 2$ in (14)] and a theoretical reflection coefficient for normal incidence of $R(0) = 10^{-4}$ [see (15)]. Fig. 1 shows a comparison of the numerical reflection coefficient with the theoretical reflection coefficient as a function of the angle of incidence θ. Similar conclusions to those obtained in [3] may be drawn from this matching.

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